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THE LOGICAL SKELETON OF ELEMENTARY DYNAMICS.[1]

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The object of this article is to outline, in as compact and time-saving a form as possible, the logical structure of elementary dynamics. The definitions and theorems here stated are believed to contain all the theoretical equipment necessary for attacking problems in elementary dynamics, up to and including problems in the general motion of a system of rigid bodies in a plane.

Although the logical sequence may not always be the best teaching order, it is hoped that the article may be of service not only to teachers but also to students, especially in reviewing the subject.[2]

I. FUNDAMENTAL CONCEPTS.

Every logical science must begin with certain undefined concepts (in terms of which all the other concepts of the science are expressed), and certain unproved propositions (from which all the other propositions of the science are deduced).

Let us begin our inquiry, therefore, by asking what are the undefined concepts of dynamics. I take them to be these: (1) Space and time, with the derived concepts of velocity and acceleration; (2) forces, as suggested by the tension and compression in our own muscles; and (3) inert material bodies, on which our forces act. Let us briefly examine each of these preliminary notions.

Acceleration. The notions of time and space, as far as, and including, the idea of velocity, may be assumed as fairly familiar to the student beginning mechanics; the idea of acceleration, however, is likely to be new, and no pains should be spared to make this concept vivid and real in the student's mind. In my own experience, a diagram showing a succession of snap shots of the moving object, taken at equal intervals of time against a stationary background, has proved very useful. Thus, in Fig. 1, as any boy can see, the body is moving to the right with an increasing velocity; while in Fig. 2, the velocity first increases and then decreases again. All the four or five important types of accelerated motion can be illustrated in this way. But the matter is at best a difficult one, and any teacher who can devise a really successful method for making acceleration seem as definite and tangible a thing as velocity, would be doing us all a great service.

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[1] This article contains the substance of a paper read before the MATHEMATICAL ASSOCIATION OF AMERICA at its first summer meeting, September 1, 1916, under the title: The Teaching of Elementary Mechanics.

[2] Any teacher who is interested is requested to communicate with the Managing Editor of the MONTHLY, as arrangements may be made for supplying reprints. The article will be held in type for a limited period, until the possible demand for reprints has been ascertained.

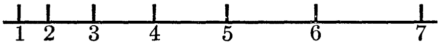

FIG. 1.

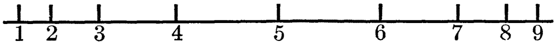

FIG. 2.

The units of measurement for acceleration, such as the railroad engineers' unit of a mile-per-hour per second, present no difficulty after the concept itself is clear.

The student should know enough calculus to understand how to get the velocity and the displacement from a known acceleration by integration. The following simple device will suffice for many common cases: If $dv/dt = f(t)$ or $f(x)$, *multiply through by* dt *or by* dx; in the latter case noting that $dx/dt = v$.

Force. The notion of force is also more or less familiar in the form of a push or a pull. We exert a force when we throw a stone or carry a load. Moreover, a simple instrument for measuring forces is also familiar, in the form of a spring balance. To find how hard I am pulling when I hold a kite string, I have merely to tie a spring balance to the end of the string and note how far the spring is stretched. Or, to find how much pressure is required to crack a nut, I have merely to replace the nut by a compression-balance between the jaws of the cracker, and note how far the spring is compressed. Of course, a spring balance, like every other physical instrument, is subject to numerous instrumental errors, and has to be somewhat idealized in order to serve our logical purpose. Nevertheless, a spring balance provides the best available concrete notion of a force.

The unit of force is perfectly arbitrary, being simply any agreed-upon amount of stretch in a standard spring; and a complete scale of multiples and submultiples of the unit is readily established by simply opposing one or more unmarked springs, in various combinations, against the standard, or unit, spring, and marking the positions reached by the pointer. (This process involves no knowledge of Hooke's Law; it assumes merely that the elastic properties of the spring do not vary with the time.)

By means of a portable, graduated spring balance thus constructed we are able theoretically to measure any force, anywhere, in terms of the arbitrarily chosen unit of force represented by our originally chosen standard spring. This simple, concrete notion of what a force *means* should not be complicated, at this stage, by any discussion of what a force *will do*, or by any inquiry as to how the ultimate unit of force can best be preserved for posterity. The important thing for the student to understand is that, theoretically at least, a force is always measurable by a spring balance, just as temperature is measurable by a thermometer, or time by a clock. Details of construction and standardization of the instrument should be postponed until later.

The following definitions will be important during the course:

(*a*) The *component, or resolved part, of a force* along a fixed axis is the magnitude of the force times the cosine of the angle which it makes with the axis.

(*b*) In a given plane, the *moment of a force* about a fixed point O in the plane is the magnitude of the force times the perpendicular distance from O to the line of action of the force. (The definition of the moment of a force about a skew axis in space need not be given until later in the course.)

As a matter of notation it is convenient to represent known forces by arrows with large closed heads, and unknown forces by arrows with large open heads; also to indicate the positive direction along a fixed axis by a small feathered arrow and the positive direction of rotation in the plane by a small, curved, feathered arrow.

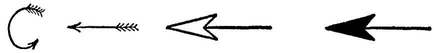

FIG. 3.

Matter. The last of our undefined concepts, namely the notion of inert material bodies, is also sufficiently familiar, provided we take it just as it comes, in its naïve, original form, and do not attempt to endow our material bodies with all sorts of mystical properties like "forces of inertia." A material body is passive. Force is the active agent, by which dead matter is buffeted about according to our will. All that we really need to know about matter, at the start, is that any given piece of matter, or any given collection of pieces, may be supposed to preserve its identity throughout any given discussion.

These, then, are our fundamental concepts: length and time, as measured by a meter stick and a clock; force, as measured by a spring balance; and inert lumps of matter, upon which our forces act.

Let us now turn to the fundamental principles, that is, the unproved propositions, of the theory.

II. FUNDAMENTAL PRINCIPLES OF THE DYNAMICS OF A SINGLE PARTICLE.

To simplify the treatment, let us begin with the study of bodies which can be treated as particles. A particle is any material body which, for the particular purpose in hand, may be regarded as concentrated at a single point. Moreover, let us assume for the present that all our observations are referred to a fixed frame of reference.

The fundamental question of dynamics is then the following: If a force gets hold of a free particle, and proceeds to act on it, what happens to the particle? The answer is contained in the first of our fundamental principles, namely:

1. **The principle of force and acceleration.** *A free particle, when acted on by a force, acquires an acceleration in the direction of the force; furthermore, if a given particle is acted on at different times by two forces* F *and* F', *and if* a *and* a' *are the corresponding accelerations, then*

$$F/F' = a/a';$$

that is, the accelerations are proportional to the forces.

This is what I like to describe as the fundamental equation of dynamics. It is best regarded as a scientific hypothesis, the truth of which has been abundantly verified by experiment. But its truth is not by any means obvious. For example, the immediate corollary that if *no force* is acting on a body, then there is *no change in the velocity* of the body, seems at first sight to contradict our commonest experience of motions which appear to die down of themselves. No pains should be spared to make clear by numerous concrete examples, including especially examples involving frictional forces, the full meaning of this equation. Each teacher will devise his own illustrations. All that I can do here is to point out that this principle of proportionality is really the kernel of the whole matter. However well a student may have done in statics, he cannot be said to have made even a fair start in dynamics until he has learned to associate force and acceleration inseparably in his mind.

Let us see what this principle enables us to do. We are on a desert island. We have our spring balance and our clock, and we are curious to discover the dynamical properties of each of the several inert lumps of matter that we find on the island. How do we proceed? Having selected any particular body for study, the essential thing is to ascertain *by experiment* (directly or indirectly) what acceleration would be produced in that body by some known force. Suppose, for example, that the given body proves to be such that a force of 10 lbs. gives it an acceleration of 2 ft./sec^2. Then the fundamental proportion enables us to state at once exactly what acceleration any other force would produce in that body.

Note on inertia. Since for any given body the ratio of the force to the acceleration produced is constant, the value of this ratio, F/a, is a characteristic of the body, which may be called its *inertia.* A body which requires a force of 10 lbs. to give it an acceleration of 2 ft./sec^2. has an inertia of 5 lbs. per (ft./sec^2.). The unit of inertia is thus a compound unit, namely a unit of force divided by a unit of acceleration; but as we shall see presently, this unit is one that seldom need be used in practice. The important thing to remember is that a separate experiment is required to determine the inertia of each individual body, and that when the inertia of any given particle is known, then the behavior of that particle under any given force can be predicted by the aid of the fundamental equation.

Up to this point we have tacitly assumed that our particle is acted on by only one force at a time. Suppose now that the particle is acted on by several forces simultaneously.

To cover this case we state a second and a third fundamental principle, as follows:

2. **The principle of the vector addition of forces,** which may be analyzed into two parts, as follows:

(*a*) *Two forces acting in the same line on the same particle may be replaced by a single force which is their algebraic sum.*

This part (*a*) of principle 2 is all that we need, besides principle 1, for the dynamics of a single particle in one dimension.

(*b*) *Two forces, acting at right angles to each other on the same particle, may be replaced by a single force which is equal to the diagonal of the rectangle formed by the given forces.*

3. **The principle of the independence of two perpendicular forces.** *Suppose a particle* P *is acted on simultaneously by two perpendicular forces,* F_x *and* F_y*, one of which always remains parallel to the* x *axis, and the other always parallel to the* y *axis. Then the motion of the projection,* P_x*, of* P *on the* x *axis will be the same as if* F_x *were the only force acting; and the motion of the projection,* P_y*, of* P *on the* y *axis will be the same as if* F_y *were the only force acting.*

These principles are not wholly independent, since either 2(*b*) or 3 can be derived from the other by the aid of the principle of force and acceleration; on account of their simplicity, however, it seems preferable to assume them both.

From these principles it follows that if a particle P is acted on by any forces in the plane, the sum of the components of the forces along the x axis will determine the component of the motion along the x axis, while the sum of the components of the forces along the y axis will determine the component of the motion along the y axis. In other words, the general problem of the motion of a particle in a plane reduces to two simpler problems, each of which can be solved by the one-dimensional theory.

The extension of the method to the motion of a particle in three dimensions requires no new principles.

III. Weight and Falling Acceleration.

The theory, as so far developed, is entirely general, so general, in fact, as to appear rather abstract to beginners. It is desirable therefore to "come down to earth" as soon as possible, and see how the fundamental equation looks when applied to the simplest case of terrestrial force, namely, the force of gravity, or weight.

Definition. The *weight,* W, of a body, in a given locality, with respect to a given frame of reference, is best defined as the force required to support the body at rest with respect to that frame in the given locality.

Thus, if I plant a tripod in any locality on the earth (or on Mars), and suspend a kettle by a spring balance underneath the tripod, the reading of the spring balance gives immediately the local weight of the kettle with respect to the earth (or to Mars); all the measurements being made, of course, in terms of the standard unit of force.

Closely connected with the weight of the body is the "falling acceleration" of the body.

Definition. The *falling acceleration* of a body, in a given locality, with respect to a given frame of reference, is best defined as the acceleration with which the body, initially supported at rest in that locality, would begin to fall (in vacuo) if the supporting thread were broken—this acceleration being measured with respect to the given frame of reference.

I have stated these two definitions in a somewhat general form, so that they may apply without change to the case of a moving frame of reference.[1] We now have the following theorem, which holds true for either fixed or moving axes:

THEOREM. *If W is the weight of a given body, in a given locality, and g is the falling acceleration of that body in the same locality, then the ratio W/g is independent of the locality, and is a correct expression for the inertia of the body.*

The proof for the case of fixed axes follows immediately from principles 1 and 2*a*. The proof for the case of moving axes belongs later in the course.

By the aid of this theorem (for fixed axes), our *fundamental equations for the motion of a single particle in a plane* can be written in the useful form

$$F_x = (W/g)\ddot{x}, \qquad F_y = (W/g)\ddot{y},$$

where F_x is the sum of the components of the forces along the x axis,
F_y is the sum of the components of the forces along the y axis,
$\ddot{x}$, $\ddot{y}$ are the second derivatives of the x and y coördinates of the particle with respect to the time,
W is the weight of the particle in any locality,
and g is the falling acceleration of the particle in that locality.

The particular advantage of using the expression W/g for the inertia of a particle, instead of the general expression F'/a', is due to the following theorem:

THEOREM. *In any given locality, the falling accelerations of all bodies are equal.*

This theorem can be proved from general considerations; or, if preferred, it may be accepted as an empirical fact.

The locality in which $g = g_0 = 980.665$ cm./sec^2. $= 32.1740$ ft./sec^2. has been adopted by international agreement as the "standard locality," and the weight of the body in that locality is called the *standard weight* of the body. Since, now, the standard value of g is the same for all bodies, we see that *the inertia of a body is immediately determined when we know its standard weight.* For example, if the standard weight of a body is 3 lbs., we know that a force of 3 lbs. will give the body the standard acceleration g_0. In practice, therefore, we seldom need refer to the inertia of a body by its original compound name; we speak simply of a *3 lb. body,* or a body whose standard weight is 3 lbs. By so doing we link up the abstract general theory with common commercial terminology, without any loss of scientific precision.

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[1] First stated in this form in *Science*, July 30, 1915, page 161.

Any two bodies which balance each other on a beam balance have obviously the same local weight, and hence, by inference, the same inertia. It should be noted, however, that the process of balancing two bodies on a beam balance gives us no information about the inertia of either body by itself; it tells us merely that the inertia of one, whatever it may be, is the same as the inertia of the other.

Note on "mass" and "quantity of matter." A lump of matter whose standard weight is 3 lbs. is often spoken of as a *mass of three pounds;* or again, such a body is often said to have a mass of three pounds, or to *contain three pounds of matter.* Thus if we are told that a certain body "has a mass of 3 pounds," we are to understand that for that body W (in the standard locality) $= 3$ lbs. Similarly for the kilogram, etc.

IV. ACCELERATIONS ALONG THE TANGENT AND NORMAL.

In writing the differential equations of motion, the axes of reference may be chosen at pleasure, provided they remain fixed throughout the discussion. It is often convenient to take as these fixed axes the tangent and normal to the path of the particle at some fixed point, Q, of the path. Then, at the instant when the particle is passing through Q, its acceleration-components along the tangent and normal have the values

$$a_T = dv/dt, \qquad \text{and} \qquad a_N = v^2/r,$$

where v is the "path-velocity" of the particle at the instant in question, and r is the radius of curvature of the path at the point Q.

In case the path of the particle is a circle of radius r,

$$a_T = r\dot{\omega} \qquad \text{and} \qquad a_N = r\omega^2,$$

where $\omega = d\theta/dt$ is the angular velocity, in radians per unit of time, and $\dot{\omega} = d\omega/dt$ is the angular acceleration. All these expressions are of great importance.

The existence of the acceleration along the normal is apt to surprise and perplex the beginner, and no pains should be spared to make the proof convincing. One method is the following: Let Fig. 4 represent snap-shots of the

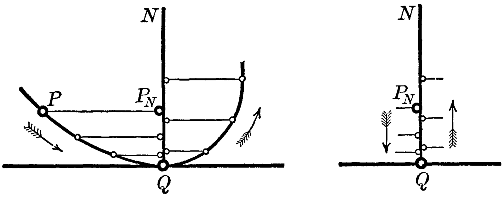

FIG. 4.

moving particle P, and let P_N be the projection of P on the normal axis. Then, by the acceleration-component of the original particle P along the normal axis, we mean simply the actual acceleration of the point P_N along that axis. Now if we draw a snap-shot picture of the point P_N, we see at once that P_N has a finite acceleration in the direction QN throughout the whole neighborhood of the instant at which it passes through Q. The existence of the normal acceleration being thus made evident, its magnitude can be readily computed by familiar methods of kinematics.

We mention also the expressions for acceleration along and perpendicular to the radius vector in polar coördinates:

$$a_r = \frac{d}{dt}\left(\frac{dr}{dt}\right) - r\omega^2, \qquad a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\omega).$$

V. FUNDAMENTAL PRINCIPLES OF THE DYNAMICS OF A SYSTEM OF PARTICLES.

In the preceding sections, we have dealt only with the motion of a single particle. In dealing with a system of particles (and we shall suppose that every material body is composed of particles) it is important to distinguish between *the external and the internal forces* of the system. In regard to the external forces, that is, those which are impressed on the system from without, we should note that each force has its own definite point of application. In regard to the internal forces, we require a fourth and last fundamental principle, as follows:

4. **Principle of action and reaction.** *When two particles are in contact with each other, or attract or repel each other according to any law like that of gravitation or magnetism, the interaction between them may be represented by a pair of twin forces, equal in magnitude and opposite in direction—one of the twins acting on one particle and one on the other, along their joining line.*

In brief, the principle of action and reaction asserts that the internal forces of a system occur in pairs of twin forces.[1]

Two definitions are important at this point:

Definition. *Center of mass.* The centroid, or center of mass,[2] of a system of particles in a plane, is a point, $(\bar{x}, \bar{y})$, such that at every instant $\bar{x}$ is the weighted average of the x's of the several particles, and $\bar{y}$ the weighted average of the y's of the several particles; that is,

$$\bar{x} = \frac{x_1w_1 + x_2w_2 + \cdots + x_nw_n}{w_1 + w_2 + \cdots + w_n}$$

---

[1] If we think of the material universe as a whole, it must be admitted that all forces are internal forces, and hence that all forces occur in pairs; but in every practical problem, what we do first is to "isolate" some definite portion of the universe; that is, we pass a knife, as it were, around the boundary of that portion, dissecting each pair of twin forces that acts across the boundary into its two constituent forces, one of which we regard as an "external force" acting on our portion, the other of which we ignore. Whether a given force shall be called an external or an internal force depends entirely on what portion of the universe we are "isolating."

[2] The term "center of mass" is intended to suggest merely the center of the material of which the body is composed. A better term might be the "center of matter." The more common term "center of gravity" is apt to be misleading.

or in case of a continuous body,

$$\bar{x} = \frac{\int x dw}{\int dw},$$

(with similar expressions for $\bar{y}$), where $w_1, w_2, \cdots$ are the standard weights of the several particles. It is easily shown that the position of this point with respect to the material system is independent of the choice of axes.

Theorem on the motion of the center of mass. *If, in a plane, a system of particles, of total weight W, is acted on by any forces, then, at any instant,*

$$F_x = (W/g)\ddot{\bar{x}} \qquad \textit{and} \qquad F_y = (W/g)\ddot{\bar{y}},$$

where F_x is the sum of the components of the external forces along OX,
F_y is the sum of the components of the external forces along OY,
and $\ddot{\bar{x}}$, $\ddot{\bar{y}}$ are the acceleration components of the associated point which we have called the center of mass.

This theorem enables us to determine the motion of the center of mass of a system without any use of the internal forces.

The proof of the theorem consists simply in writing the differential equation of motion for each particle (taking account of both the external and the internal forces that act on that particle), and adding these equations; the internal forces will be found to cancel out by pairs.

The extension to three dimensions is obvious.

Definition. *Radius of gyration about a given axis.* Suppose a body is composed of particles whose distances from a given axis, fixed in the body, are $r_1, r_2, \cdots, r_n$. Then the radius of gyration of the body about that axis is a distance, k, such that the square of k is the weighted average of the squares of the r's of the several particles; that is,

$$k^2 = \frac{r_1^2 w_1 + r_2^2 w_2 + \cdots + r_n^2 w_n}{w_1 + w_2 + \cdots + w_n},$$

or, in case of a continuous body,

$$k^2 = \frac{\int r^2 dw}{\int dw}.$$

The following properties of the radius of gyration follow readily from the definition:

(*a*) If k is the radius of gyration about any axis, and $\bar{k}$ the radius of gyration about a parallel axis through the center of mass, then $k^2 = \bar{k}^2 + a^2$, where a is the distance between the axes.

(*b*) In case of a lamina in the x-y plane, if k is the radius of gyration about an axis perpendicular to the plane at the origin, then $k^2 = k_x^2 + k_y^2$, where k_x and k_y are the radii of gyration about the x and y axes, respectively.[1]

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[1] The radius of gyration, k, seems to me a more tangible and useful quantity in dynamics than the so-called "moment of inertia." If you ask a student to estimate by eye the moment of

The student should know enough of the calculus to be able to find the center of mass and radius of gyration of all the simpler bodies by integration.

We are now in position to study the general problem of the motion of a rigid body in a plane.

First, we define the *angular acceleration*, $\ddot{\theta}$, or $\dot{\omega}$, of a rigid body in a plane as the second time-derivative of the angle θ which a line fixed in the body makes with a line fixed in the plane. Then we establish the following theorem:

Theorem of rotation, or Theorem of moments about a point fixed in the body. *If a rigid body, free to move in any manner in the plane, is acted on by any forces, and if* Q *is any suitably chosen point fixed in the body, then, at any instant,*

$$Fp = (W/g)k^2\dot{\omega},$$

where $\dot{\omega}$ *is the angular acceleration of the body at the instant in question,* Fp *is the sum of the moments of the external forces about* Q *at that instant, and* k *is the radius of gyration of the body about* Q*. Here any point* Q*, fixed in the body, is a suitable point about which to take moments, provided either:* (*a*) Q *is fixed in the plane* (*this is the case of rotation about a fixed axis*)*; or* (*b*) Q *coincides with the center of mass; or* (*c*) Q *has a vector acceleration the direction of which passes through the center of mass.*

This theorem enables us to determine the angular acceleration of the body without any consideration of the internal forces.

The proof of the theorem for case (*a*) consists essentially in setting up a differential equation of motion for each particle separately, taking account of both internal and external forces, and choosing the axis along the tangent to the path in each case. Then, after multiplying each equation by a suitable factor and adding, we find that the internal forces will cancel out by pairs. The proof for the other cases follows without difficulty.[1]

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inertia of a baseball bat about an axis through one end, he will be entirely at a loss to reply. He probably will not know even the name of the unit in which moment of inertia is measured. But if you ask him to estimate the radius of gyration of the bat about one end, he will see at once that something like 2 feet would be a plausible answer. The ability to give an approximate estimate of the numerical magnitude of the quantities with which one has to deal seems to me a very important consideration in favor of the use of radius of gyration instead of moment of inertia. Furthermore, the statement of theorem (*a*), above, is decidedly simpler in terms of radius of gyration than it would be in terms of moment of inertia. A similar remark applies to the following theorems:

(*c*) If a physical pendulum is swung about a fixed axis, then $k^2 = al$, where k is the radius of gyration of the pendulum about the axis, a = the distance from the axis to the center of mass, and l = the length of the isochronous simple pendulum.

(*d*) If a submerged vertical plane area serves as a dam, then $k^2 = \bar{x}p$, where k = the radius of gyration of the area about the water line, $\bar{x}$ = the distance from the water line to the center of mass of the area, and p = the distance from the water line to the center of pressure.

Neither of these theorems could be so simply stated in terms of moment of inertia.

Finally, the inappropriateness of the term "moment" should be noted; for moment of inertia is not the product of an inertia and a lever arm. If the concept is to be used in dynamics at all, the term "spin inertia," suggested by Professor W. S. Franklin to denote the quantity Wk^2, is recommended.

[1] For one method of proof, see a paper by the present writer in the AMERICAN MATHEMATICAL MONTHLY, Vol. 21, pp. 315–320, December, 1914.

An easy way to recall to mind the form of this important equation is the following: consider the motion of a single particle, attached to a fixed axis by a weightless arm, k, and acted on by a single force at right angles to the arm; then write the differential equation of motion along the tangent to the path, and note that $v = k\omega$.

These two theorems, the theorem of the center of mass and the theorem of rotation, bring us nearly to the end of our inquiry. For the principles and theorems here set forth, small as their number may seem, are sufficient for the solution of any elementary problem in the motion of a rigid body in two dimensions, as far as the dynamics of the problem is concerned. The only difficulties which remain are connected with the solution of the differential equations of motion; these difficulties are of a purely mathematical sort, and will not be considered here. Even for problems in three dimensions, no new dynamical principles are required, the only complications which arise being chiefly of a kinematical nature.

No discussion of dynamical principles would be complete, however, without mention, at least, of the useful theorem of work and kinetic energy, and the somewhat less important theorems of impulse and momentum (linear and angular). These theorems, while not essential for the theoretically complete solution of a problem, will often yield certain partial results of great value, with much less labor than the use of the general equations of motion would require.

VI. WORK AND KINETIC ENERGY.

Definition. *The work done by a force F, during a given interval* from t_1 to t_2, is defined as the quantity

$$U = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_z dz, \qquad \text{or} \qquad U = \int_{s_1}^{s_2} F_T ds,$$

where F_x, F_y, F_z are the components of F along the axes, and dx, dy, dz refer to the projections on the axes of the particle P on which F acts; or, where ds is the element of arc of the path traced by P, and F_T is the component of F along the tangent to the path.

In brief, *Work = Force × Distance.*

In applying this definition, it is important to note that every force must have a definite particle P as its point of application.

Definition. *The kinetic energy of a particle w, at a given instant,* is defined as the quantity

$$\text{K.E.} = (w/g)(v^2/2),$$

where v is the path velocity of the particle at that instant.

To recall these definitions to mind, for the case of a constant force, we have merely to note the result of integrating the fundamental equation $F = (W/g)dv/dt$ with respect to x; the left side gives work, the right side, change in kinetic energy.

Theorem of work and kinetic energy. *In any system of particles, the total* WORK DONE *by all the forces (both external and internal), during any interval, is equal to the total* GAIN IN KINETIC ENERGY *during that interval. But in the case of a rigid body, the work done by the internal forces is zero,* so that in applying the theorem to the case of a rigid body, only the external forces need be considered.

Note 1. In case of a rigid body rotating about a fixed axis (fixed in the body and fixed in space), the kinetic energy at any instant is $(W/g)k^2(\omega^2/2)$, where k is the radius of gyration of the body about that axis, and ω is the angular velocity of the body. In other words, if a rigid body is rotating about a fixed axis, its kinetic energy is the same as if all its material were concentrated at a distance k from that axis.

Note 2. In the case of a rigid body moving in any manner in a plane, the total kinetic energy, at any instant, is given by

$$(W/g)(\bar{v}^2/2) + (W/g)\bar{k}^2(\omega^2/2),$$

where $\bar{v}$ is the path velocity of the center of mass at the given instant,
ω is the angular velocity of the body at that instant,
and $\bar{k}$ is the radius of gyration of the body about the center of mass.

The first term of this expression is the *translational kinetic energy* (as if all the material were concentrated at the center of mass); the second term is the *rotational kinetic energy* (as if the center of mass were fixed).

VII. IMPULSE AND MOMENTUM.

Definition. The *impulse of a force F during a given interval* from t_1 to t_2, is defined as a vector quantity whose components along the axes of reference are

$$\int_{t_1}^{t_2} F_x\,dt, \qquad \int_{t_1}^{t_2} F_y\,dt, \qquad \int_{t_1}^{t_2} F_z\,dt,$$

where F_x, F_y, F_z are the components of F along the axes.

In brief, *Impulse = Force × Time.*

Since impulse (unlike work) is a vector quantity, we may speak of the component of an impulse along any fixed line, and of the moment of an impulse about any fixed axis.

Definition. The *momentum of a particle w at a given instant* is defined as a vector quantity whose components along the axes are

$$(w/g)v_x, \qquad (w/g)v_y, \qquad (w/g)v_z,$$

where v_x, v_y, v_z are the components of the path velocity v of the particle at that instant.

Since momentum (unlike kinetic energy) is a vector quantity, we may speak of the component of a momentum along any fixed line, and of the moment of a momentum about any fixed axis.

To recall these definitions to mind, for the case of a constant force, we have only to note the result of integrating the fundamental equation $F = (W/g)dv/dt$ with respect to t; the left side gives impulse, the right side change in momentum.

Theorem of linear impulse and momentum. *In any system of particles, the total* IMPULSE *of all the external forces, in any given direction, during any interval, is equal to the total* GAIN IN MOMENTUM *in that direction during that interval.*

Note. In the case of any system of particles, whether rigidly connected or not, the total momentum in the x-direction, at any instant, is given by

$$(W/g)\bar{v}_x,$$

where $\bar{v}_x$ is the velocity component of the center of mass in that direction. In fact, the theorem of linear impulse and momentum is merely a corollary of the theorem on the motion of the center of mass.

Theorem of angular impulse and momentum, or **Theorem of moments about an axis fixed in space.** *In any system of particles, the total* MOMENT OF IMPULSE *of all the external forces about any fixed axis, during any interval, is equal to the total* GAIN IN MOMENT OF MOMENTUM *about that axis, during that interval.*

Note 1. In case of a rigid body rotating about a fixed axis (fixed in the body and fixed in space), the moment of momentum about that axis at any instant is $(W/g)k^2\omega$, where k is the radius of gyration of the body about that axis, and ω is the angular velocity of the body. In other words, if a rigid body is rotating about a fixed axis, its moment of momentum about that axis is the same as if all its material were concentrated at a distance k from the axis.

Note 2. In the case of a rigid body moving in any manner in a plane, the total moment of momentum about any point O fixed in the plane, at any instant, is

$$(W/g)p\bar{v} + (W/g)\bar{k}^2\omega,$$

where ω is the angular velocity of the body at the instant in question,
$p\bar{v}$ is the moment about O of the path velocity of the center of mass at that instant,
and $\bar{k}$ is the radius of gyration of the body about its center of mass.

The first term of this expression is the moment of the *translational momentum* of the body (as if all the material were concentrated at the center of mass); the second term is the moment of the *rotational momentum* of the body (as if the center of mass were fixed).

VIII. Note on the Fundamental Principles of Statics.

Problems in statics can always be regarded as special cases of problems in dynamics in which the velocity is zero, so that the theory of statics is completely covered by the theory of dynamics. If, however, one wishes to study statics by itself, before taking up the study of dynamics, it is interesting to note that the following four fundamental principles are sufficient as a logical foundation for statics.

A. Principle of the vector addition of forces [see 2, above].

B. Principle of the transmissibility of forces in a rigid body: Any force acting on a rigid body may be shifted at pleasure along its line of action.

By the aid of these principles *A* and *B*, any set of forces acting on a rigid body can be " boiled down " either to a single force, or else to a single couple (that is, a pair of equal forces acting in opposite directions along parallel lines). This force, or couple, is called the *resultant* of the given set of forces.

C. Principle of static equilibrium: Suppose a rigid body is acted on by a set of forces which " boil down," by principles *A* and *B*, to zero; then if the body was initially at rest (with respect to a fixed frame of reference), it will remain at rest (with respect to that frame).

D. Principle of action and reaction between two bodies [see 4, above].

The concepts of acceleration and inertia are not required in statics.

IX. Systems of Derived Units.

The units of measurement employed in mechanics divide themselves into (1) units for the fundamental quantities, force, length, and time; and (2) units for the derived quantities, such as velocity, acceleration, work, energy, etc.

The fundamental units may be chosen at pleasure. The derived units are defined in a systematic manner, in order to avoid much unnecessary labor in computation. For example, if the unit of length is the foot, the systematic unit of area is the square foot (ft^2.), not the acre.

The fundamental units of force, length, and time which are adopted as the basis of any system of units must of course be so defined as to be capable of exact reproduction at any future time; and these definitions, uninteresting as they properly are to the beginner, form an essential part of the logical skeleton of the science. Thus:

The *kilogram force* (1 kg.) is defined as the force required to support a certain carefully preserved lump of metal called the " international standard kilogram," in vacuo, in the standard locality. (Here the " standard locality," as fixed by the International Bureau of Weights and Measures, Paris, 1901, means any locality where $g = g_0 = 980.665$ cm./sec^2. = 32.1740 ft./sec^2., this being approximately the value of g at 45° latitude, sea level.) The *dyne* = 1/980665 kg.

The *pound force* (1 lb.) is defined as the force required to support another carefully preserved lump of metal, called the " standard pound avoirdupois," in vacuo, in the standard locality. (1 lb. = 0.4535924 $\cdots$ kg.)

The *meter length* (1 m.) is defined as the distance, at the temperature of melting ice, between two scratches on a certain carefully preserved metal bar, called the " international standard meter." The *foot* (1 ft.) is defined, in the United States, in terms of the meter, through the relation 1 meter = 39.37 inches (where 12 inches = 1 ft.). The *centimeter* (1 cm.) = 0.01 m.

The *second of time* is defined as 1/86400 of a mean solar day.

The three most important systems of units are the lb.-ft.-sec. system, the kg.-m.-sec. system, and the dyne-cm.-sec. system (the last being commonly called the " c.g.s." system). The names of the principal derived units in each of these three systems are shown in Table I, where the symbol / is to be read " per."

TABLE I.

PRACTICAL SYSTEMS OF DERIVED UNITS, AS ACTUALLY USED BY ENGINEERS AND PURE SCIENTISTS.

Dynamical Quantity.	Derivation.	Name of Unit in the Practical British System.	Name of Unit in the Practical Continental System.	Name of Unit in the Practical "C. G. S." System.	Dimensions in Terms of F, L, T.
FORCE	Fundamental	lb.	kg.	dyne	F
Length	Fundamental	ft.	m.	cm.	L
Time	Fundamental	sec.	sec.	sec.	T
Velocity	Length per unit time	ft./sec.	m./sec.	cm./sec.	L/T
Acceleration	Velocity per unit time	ft./sec^2.	m./sec^2.	cm./sec^2.	L/T^2
Pressure	Force per unit area	lb./ft^2.	kg./m^2.	dyne/cm^2.	F/L^2
Impulse, or Momentum	Force × time	lb.-sec.	kg.-sec.	dyne-sec.	FT
Work, or Kin. energy	Force × dist.	ft.-lb.	kg.-m.	dyne-cm.	FL
Power	Force × veloc.	ft.-lb./sec.	kg.-m./sec.	dyne-cm./sec.	FL/T
Inertia	Force per unit accel.	lb. per (ft./sec^2.) [= 1 "slug"]	kg. per (m./sec^2.) [= 1 "metric slug"]	dyne per (cm./sec^2.) [= 1 "Gram mass"]	$F/(L/T^2)$

All the units in this table, except the units of inertia, are in everyday practical use, the first two systems being preferred by engineers, the third by physicists.[1] The units of inertia, as already explained, are seldom required, the inertias, or masses, of two bodies being most readily compared by means of their standard weights as determined by weighing on a beam balance. The commonly used units of mass are the *pound mass* and the *kilogram mass;* such systematic units as the "slug" (= 32.1740 pounds mass), and the "metric slug" (= 9.80665 kilograms mass) are never used in practice.

APPENDIX.

In spite of the simplicity and familiarity of the systems of units shown in Table I, most text-books make the attempt to impose upon the student's attention a different and unpractical group of systems, namely, systems in which the derived units are based on *mass*, length, and time, instead of on *force*, length, and time. That this attempt has not been successful outside of the text-books is sufficiently shown by Table II, where the names of the units systematically derived from mass, length, and time are given in a form in which they may be readily contrasted with the units of Table I. It will be seen that not one of the units of pressure, work, power, etc., named in Table II is in practical use, either in engineering or in pure science. It is earnestly to be hoped that all these superfluous units will eventually be dropped from the text-books.

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[1] Other units, which do not belong to any system, are also in use, and are sometimes convenient for special purposes. For example, 1 atmosphere = 14.70 lb./in.^{2}; 1 horsepower = 550 ft.-lb./sec.; 1 joule = 10^7 dyne-cm.; 1 watt = 10^7 dyne-cm./sec.; etc. Every such non-systematic unit, if used at all, should be defined explicitly in terms of one of the systematic units of the same kind.

TABLE II.

A USELESS COMPLICATION, STILL FOUND IN MANY TEXT-BOOKS.

Dynamical Quantity.	Derivation.	Name of Unit in the Scholastic British System.	Name of Unit in the Scholastic Continental System.	Name of Unit in the Scholastic “C. G. S.” System.	Dimensions in Terms of M, L, T.
MASS	“Fundamental”	Lb.	Kg.	Gm.	M
Length	Fundamental	ft.	m.	cm.	L
Time	Fundamental	sec.	sec.	sec.	T
Velocity	Length per unit time	ft./sec.	m./sec.	cm./sec.	L/T
Acceleration	Velocity per unit time	ft./sec^2.	m./sec^2.	cm./sec^2.	L/T^2
Pressure	Mass-accel’n per unit area	Lb. per (sec^2.-ft.)	Kg. per (sec^2.-m.)	Gm. per (sec^2.-cm.)	$M/(T^2L)$
Momentum, cr Impulse	Mass × veloc.	Lb.-ft./sec.	Kg.-m./sec.	Gm.-cm./sec.	ML/T
Kin. energy, or Work	Mass × (veloc.)^2	Lb.-ft^2./sec^2.	Kg.-m^2./sec^2.	Gm.-cm^2./sec^2.	ML^2/T^2
Power	Mass-(veloc.)^2 per unit time	Lb.-ft^2./sec^3.	Kg.-m^2./sec^3.	Gm.-cm^2./sec^3.	ML^2/T^3
Force	Mass × accel’n	Lb.-ft./sec^2. = 1 “poundal”	Kg.-m./sec^2.	Gm.-cm./sec^2. = 1 “dyne”	ML/T^2

A comparison of these two tables—Table I simple and familiar, Table II strange and artificial—provides one of the best arguments in favor of the use of force rather than mass as the principal undefined concept of dynamics. The only reason why the text-books so insistently base their derived units on mass instead of on force is apparently that a standard lump of metal is easier to preserve in a museum than a standard spring balance. But this is no argument for the logical priority of mass over force. As a matter of fact, the fundamental unit of force is as easy to preserve as the fundamental unit of mass, though the method of doing so does not consist in simply storing away a spring balance. Every consideration of logical simplicity and practical convenience speaks in favor of Table I as against Table II.

Finally, it should be noted that it is not possible to take *both* force *and* mass as fundamental units, without giving up the whole idea of systematic derived units. Every attempt to treat the units of force and mass as equally fundamental proves to be really a definition of one of them in terms of the other, so that the necessity of choosing between Table I and Table II cannot in that way be escaped.[1]

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[1] This matter is closely related to the question of the choice between $F/F' = a/a'$ and $F = ma$ as the fundamental equation of mechanics. On this question reference may be made to a recent controversy between the present writer and Professor L. M. Hoskins in *Science* (December 4, 1914; February 5, April 23, July 30, and September 10, 1915; March 3, June 30, September 8, and October 27, 1916).